On Symmetries in Nonlinear Quantum Mechanics*

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Abstract

It is shown how nonlinear versions of quantum mechanics can be refolmulated in terms of a (linear) C^*-algebraic theory. Then also their symmetries are described as automorphisms of the correspondong C^*-algebra. The requirement of "conservation of transition probabilities" is discussed.

1 A formulation of nonlinear quantum mechanics

Nonlinear quantum mechanics (NLQM) is usually formulated in a form of nonlinear Schrödinger equation (NLSchE), using concepts of the traditional (linear) quantum mechanics (QM). Let $\mathcal{H}$ be the Hilbert space of QM, and the set of "pure states" is identified with the projective Hilbert space $P(\mathcal{H})$, i.e. the set of one–dimensional projections P_ψ on $\mathcal{H}$ ($P(\mathcal{H})$ is identified also with the set of "rays" $\boldsymbol{\psi} := P_\psi\mathcal{H} \ni \psi \neq 0$, i.e. to the set of one–dimensional complex subspaces $\boldsymbol{\psi} \subset \mathcal{H}$), endowed with the quotient topology induced from $\mathcal{H}$. Then the considered class of NLSchE can be formulated with a help of a selfadjoint operator valued function $\boldsymbol{\psi}\ (\in P(\mathcal{H})) \mapsto \tilde{H}(\boldsymbol{\psi}) \equiv \tilde{H}(\boldsymbol{\psi})^*$ defined on a dense subset $D(\tilde{H})$ of $P(\mathcal{H})$. Then the NLSchE is written as

$$i\cdot\partial_t\psi_t = \tilde{H}(\boldsymbol{\psi_t})\psi_t, \quad \psi_t \in D(\tilde{H}(\boldsymbol{\psi_t})). \tag{1}$$

We shall restrict our attention to such NLSchE (1) where the "Hamiltonians", i.e. the functions $\tilde{H}$ are of a specific form. To express it in a concise form, let us consider $P(\mathcal{H})$ as a submanifold of the real Banach space of symmetric trace class operators $\mathfrak{T}_s$. Then the differential (resp. Fréchet derivative) $D_\varrho h$ of a real–valued function $h : \mathfrak{T}_s \mapsto \mathbb{R}$ (in points $\varrho \in \mathfrak{T}_s$, where it is well defined) is a bounded real linear functional on the tangent space $T_\varrho P(\mathcal{H})$, if $\varrho \in P(\mathcal{H})$ (cf. the concept of $\boldsymbol{\mathcal{D}}_r$–generalized differential in [2]), and it corresponds to a symmetric linear operator.[1] We shall assume,

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[1]This correspondence uses the duality between $\mathfrak{T}_s$ and $\mathcal{L}(\mathcal{H})_s$ given by the trace of products of operators, and uses also a restriction to the submanifold $P(\mathcal{H})$; the present exposition is, however, rather simplified.

in the sense of the mentioned correspondence, that the operator–valued function $\boldsymbol{\psi} \mapsto \tilde{H}(\boldsymbol{\psi})$ corresponds to a differential $D_{\boldsymbol{\psi}} h$. This is the case of several usual nonlinear modifications and/or approximations of QM, cf. [2, 3, 4]. In such a case, the NLSchE (1) is a form of classical Hamilton equations on $P(\mathcal{H})$, where the Poisson brackets are defined as the unique extension of ones given for functions of the form $h_A(P_\psi) := Tr(P_\psi A)$ by the relation

$$\{h_A, h_B\}(P_\psi) := i \cdot Tr(P_\psi [A, B]) \equiv h_{i[A,B]}(P_\psi). \tag{2}$$

The specific case of Schrödinger equation (with the Hamiltonian $H = H^*$) of the ordinary (linear) QM is obtained from the Hamiltonian function $h := h_H$.

2 Symmetries in linear QM

A general symmetry transformation Φ of QM is any bijection $\Phi : P(\mathcal{H}) \to P(\mathcal{H})$ conserving the "transition probabilities":

$$Tr\big(\Phi(P_\psi)\Phi(P_\varphi)\big) = Tr(P_\psi P_\varphi), \quad \forall P_\psi, P_\varphi \in P(\mathcal{H}). \tag{3}$$

For continuous one–parameter groups $t \mapsto \Phi_t$ of such symmetries, the known Wigner theorem gives

$$\Phi_t(P_\psi) \equiv e^{-itH} P_\psi e^{itH}, \tag{4}$$

for some selfadjoint H, determined up to an additive numerical constant uniquely; this is, however, a flow corresponding to the linear theory with Hamiltonian H. Hence, the relation (3) cannot be satisfied in nonlinear NLQM.

It is possible, however, to reformulate (3) in a way that can be extended to NLQM. It is done by considering one of the projections in (3), say P_ψ, as representing a state ω_ψ, but the second one will be considered as an observable. The time evolution $A \mapsto \Phi_t^T(A)$ of observables A is just the rewriting of the given Schrödinger evolution $\Phi_t : \omega \mapsto \Phi_t \cdot \omega$ of states ω into the "Heisenberg picture", what is just the dual (i.e. transposed) transformation to that of states:

$$Tr(P_\psi \Phi_t^T(A)) := Tr(\Phi_t(P_\psi)A) \equiv (\Phi_t \cdot \omega_\psi)(A), \quad \forall\ \{\psi, A, t\}. \tag{5}$$

Then the condition (3) for $\Phi_t \mapsto \Phi$ has the form:

$$(\Phi_t \cdot \omega_\psi)\left(\Phi_{-t}^T(A)\right) = Tr\left(\Phi_t(P_\psi)\Phi_{-t}^T(A)\right) = Tr(P_\psi A), \tag{6}$$

what is trivially valid for any linear transformation Φ_t, and its transposed Φ_t^T. As we shall see, a natural definition of "Heisenberg picture"–like transformations Φ_t^T of observables in NLQM corresponding to a given (nonlinear) flow Φ_t on $P(\mathcal{H})$ makes (6) *valid also for nonlinear flows* on $P(\mathcal{H})$. Clearly, such a "trivialization" of the relation (3) looses its informative value: (3) implies possibility of unitary implementation of Φ_t, but (6) does not imply, perhaps, anything on the form of Φ_t.

Let us note that the *transition probabilities interpretation of* (3) can be traced back to the ***r***eduction postulate of Dirac and von Neumann for the process of measurement in QM: States P_ψ

"jump" into eigenstates P_φ of the measured observable corresponding to the obtained numerical result (being equal to the corresponding eigenvalue) with probabilities expressed by (3). If we accept these "jumps" as physical processes, the usual interpretation of (3) is the natural requirement of invariance of the probabilities of these processes with respect to any symmetry transformation, e.g. with respect to the transition to another reference frame. The proposed interpretation connected with (6), on the other hand, requires *invariance of all expectations* with respect to such simultaneous (symmetry) transformations of states, and also of observables, that are mutually connected in the above described way.

3 "Koopman–like" linear reformulation of NLQM

Let us remind first, how the classical Hamiltonian mechanics on a (2n–dimensional) symplectic manifold $(M;\Omega)$ can be (almost equivalently: up to "measure zero problems") rewritten in a linear form according to [1]: Any symplectic flow ϕ_t on M conserves the Liouville measure $\wedge^n\Omega$. If we introduce the Hilbert space $\mathcal{H} := L^2(M,\wedge^n\Omega)$, as well as the transformations

$$\big(U_t^\phi f\big)(m) := f\big(\phi_t(m)\big), \quad f \in L^2(M,\wedge^n\Omega), \tag{7}$$

then U_t^ϕ are unitary and (under some continuity condition) are expressible by a linear selfadjoint "Liouville operator" L_ϕ : $\quad U_t^\phi \equiv \exp(-itL_\phi)$. We have obtained a linear dynamical system on infinite dimensional $\mathcal{H}$ containing (up to measure zero subsets of M) all the information about the given finite dimensional (nonlinear) Hamiltonian system.

Let us now formulate, in a way "similar" to the Koopman's one, a linear quantum theory (QT) containing a given NLQM as a subtheory. Here the C^*-algebraic framework will be useful. We shall also generalize straightforwardly NLQM to a dynamics of all density matrices.

(a) "Quantum phase space" consists of all density matrices $\varrho \in \mathcal{S}_* := \mathfrak{T}_{+1} \subset \mathfrak{T}_{\mathfrak{s}}$. On $\mathcal{S}_*$, canonical Poisson brackets are defined:

$$\{f,h\}(\varrho) := i\cdot Tr\big(\varrho[D_\varrho f, D_\varrho h]\big), \quad f,h \in C^\infty(\mathcal{S}_*), \tag{8}$$

where the differentials $D_\varrho f, \dots$ are considered as operators according to the above mentioned identification: $T_\varrho^*\mathfrak{T}_s \simeq \mathcal{L}(\mathcal{H})_s$.

(b) One–parameter symmetry (evolution) groups $\Phi_t : \mathcal{S}_* \to \mathcal{S}_*$ are chosen to be the Hamiltonian flows with respect to (8). These evolutions contain also solutions of all the considered NLSchE, if restricted to $P(\mathcal{H})$.

(c) The C^*-algebra of observables is chosen to be $\mathcal{C} := C_b(\mathcal{S}_*, \mathcal{L}(\mathcal{H}))$:=the set of bounded–operator–valued functions continuous in some "convenient" topologies on $\mathcal{S}_*$ and $\mathcal{L}(\mathcal{H})$.[2]

(d) Any Hamiltonian flow Φ of (b) can be realized by a unitary cocycle $u^\Phi : \mathbb{R} \times \mathcal{S}_* \to \mathcal{U}(\mathcal{H})$ (:=

[2]Let us note that a necessity of a state–dependence of observables is a consequence of nonlinearity of transformations and of symmetry requirements of the type (6).

the unitary group of $\mathcal{H}$), $(t;\varrho) \mapsto u^\Phi(t,\varrho)$ so that

$$\Phi_t \cdot \varrho := \Phi_t(\varrho) \equiv u^\Phi(t,\varrho)\varrho u^\Phi(t,\varrho)^*. \tag{9}$$

If the flow Φ is determined by a Hamiltonian $h : \mathcal{S}_* \to \mathbb{R}$, then the corresponding cocycle u^Φ can be chosen as the unique solution of NLSchE written in the form:

$$i\partial_t u^\Phi(t,\varrho) = D_{\varrho_t} h \cdot u^\Phi(t,\varrho), \quad u^\Phi(0,\varrho) \equiv \mathbb{I}, \quad \varrho_t := \Phi_t(\varrho). \tag{10}$$

(e) The transformation Φ_t^T of "observables" $\mathfrak{f} \in \mathcal{C}$, $\mathfrak{f} : \varrho \mapsto \mathfrak{f}(\varrho) \in \mathcal{L}(\mathcal{H})$ is then defined by a quantum analogy of (7):

$$\big(\Phi_t^T \cdot \mathfrak{f}\big)(\varrho) := u^\Phi(t,\varrho)^* \mathfrak{f}\big(\Phi_t \cdot \varrho\big) u^\Phi(t,\varrho), \tag{11}$$

what defines a one–parameter (automorphism) group of (linear!) transformations of the linear space $\mathcal{C}$ of operator–valued functions on the "elementary state space" $\mathcal{S}_*$.[3]

(f) Let us consider, e.g., only the states $\omega \in \mathcal{C}^*_{+1}$ on the C^*-algebra $\mathcal{C}$ of the form

$$\omega(\mathfrak{f}) = \int_{\mathcal{S}_*} Tr\big(\varrho \mathfrak{f}(\varrho)\big)\ \mu_\omega(d\varrho), \tag{12}$$

where μ_ω is any probability measure on $\mathcal{S}_*$. The states corresponding to points $\varrho \in \mathcal{S}_*$ are represented by the Dirac measures $\mu := \delta_\varrho$. Let us define the transformation

$$(\Phi_t \cdot \omega)(\mathfrak{f}) := \big(\Phi_t^{TT} \cdot \omega\big)(\mathfrak{f}) \equiv \omega\big(\Phi_t^T \cdot \mathfrak{f}\big),$$

i.e. the transposed map of the automorphism Φ_t^T from (11). Then, according to (11) and (9), one obtains an extension of (9) to more general states:

$$(\Phi_t \cdot \omega)(\mathfrak{f}) \equiv \omega(\Phi_t^T \cdot \mathfrak{f}) = \int Tr\big(\Phi_t \cdot \varrho\ \mathfrak{f}(\Phi_t \cdot \varrho)\big)\mu_\omega(d\varrho). \tag{13}$$

Hence, also a nonlinear version of "transition robability conservation" is fulfilled:

$$Tr\Big(\Phi_t \cdot \varrho\ \big(\Phi_{-t}^T \cdot \mathfrak{f}\big)(\Phi_t \cdot \varrho)\Big) \equiv Tr\big(\varrho \mathfrak{f}(\varrho)\big). \tag{14}$$

4 Summary

Hamiltonian forms of NLQM are contained in a linear QT that is formulated in terms of a C^*-algebra $\mathcal{C}$ of operator valued functions on the space of all density matrices. Then the automorphism group of $\mathcal{C}$ contains also nonlinear symmetry groups of the considered quantum system. Although

[3]Density matrices $\varrho \in \mathcal{S}_*$ represent states of the quantum system called *elementary mixtures*. They should be distiguished from *genuine mixtures* expressed by probability measures on $\mathcal{S}_*$: The physical interpretations of different measures with the same barycentre are mutually different in NLQM.

the usual requirement of "transition probabilities conservation" leads to linear transformations of the Hilbert space, a natural reinterpretation of this requirement is extendable also to NLQM.

Let us note, that the theoretical scheme sketched above can be developed into a theory containing, as exact subtheories, besides the mentioned QM and NLQM, also CM, and also various "quasiclassical approximations" (as are WKB, or time dependent Hartree-Fock theory), cf. [2, 3]; we call this theory EQM (:= extended quantum mechanics), [2].

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